A SIMPLE STOCHASTIC MODEL OF NON-IDEAL MIXER*

V. KUDRNA, J. VLČEK and I. FOŘT

Department of Chemical Engineering,
Institute of Chemical Technology, 166 28 Prague 6

Received April 27th, 1976

A simple model is presented in the paper of an non-ideal mixer. Assuming that the process in question is random it is shown that the distribution of the residence time in the mixer may be described by the gamma distribution. The proposed model was tested experimentally with good results. An attempt is made to interpret the model physically.

Numerous attempts exist in the literature to describe the distribution of residence time in a real mixer. According to our opinion, however, very little attention was devoted to the gamma distribution. An expression for the gamma distribution function can be obtained easily by representing formally the real mixer by a cascade of ideal mixers

$$f_r(t) = \lambda \frac{(\lambda t)^{m-1}}{(m-1)!} \exp(-\lambda t) \tag{1}$$

and replacing the integer m, designating the number of ideal mixers, by an arbitrary number exceeding unity.

We have succeeded in finding only several works[4,14–17], mostly without being able to study their original content, which deal with this problem. According to our opinion this is the consequence of the lack of physical interpretation of the expression viewing the real mixer as a "cascade of a non-integer number of ideal mixers". This paper presents on one hand a simple stochastic model leading to the gamma distribution and attempts to interpret the real mixer in the above manner.

THEORETICAL

From the view point of random processes an ideal mixer may be thought of as a tank in which the residence time, T_i, is a random variable with an exponential distribution of the form

$$P\{0 \leqq T_i < t_i\} = F_i(t_i) = 1 - \exp(-\lambda t_i)$$

* Part XLV in the series Studies on Mixing; Part XLIV: This Journal *40*, 3794 (1975). In an abbreviated form presented at the V-th International CHISA Congress, Prague 1975.

and the probability density

$$f_i(t_i) = \lambda \exp(-\lambda t_i) \tag{2}$$

and the mean

$$\bar{t}_i \equiv E(T_i) = \int_0^\infty t_i f_i(t_i) \, dt_i = 1/\lambda \, . \tag{3}$$

Introducing at the inlet of the ideal mixer a concentration impulse $c_1(t) = \alpha\delta(t)$, where α is a proportionality constant and $\delta(t)$ is the Dirac function[7], we obtain at the outlet a response in the form

$$c_2(t) = \int_0^t f_i(t - t_i) \, c_1(t_i) \, dt_i = \alpha\lambda \exp(-t) \, . \tag{4}$$

From this it follows that at the outlet of the mixer there is a non-zero concentration from the very beginning. This situation, provided that the outlet does not coincide with the inlet, is clearly implausible in a real mixer.

Let us consider therefore that only a part of the real mixer is being mixed ideally and that the "ideal" residence time T_i is increased by a time lag T_p, which is also a random variable with the probability density

$$f_p(t_p) = dP\{0 \leqq T_p < t_p\}/dt_p \, .$$

The residence time in the real mixer, T_r, is thus also a random variable given by

$$T_r = T_i + T_p \tag{5}$$

with the probability density

$$f_r(t) = dP\{0 \leqq T_r < t\}/dt$$

and the mean

$$\bar{t}_r = E(T_r) = \int_0^\infty t \, f_r(t) \, dt = 1/\varkappa \, . \tag{6}$$

In order that we may characterize the way the real mixer differs from the ideal mixer, let us introduce the degree of ideality Z of the real mixer by the relation

$$Z \equiv T_i/t = Z(t) \, . \tag{7}$$

This degree indicates the "remoteness" of the real mixer from the ideal behaviour for a given value $T_r = t$. It is apparent that Z may assume values from the interval between 0 and 1.

The mean degree of ideality, $\bar{z}$, will be the expected value of the above definition equation, namely

$$\bar{z} = E\{Z(t)\} = \bar{z}(t) . \tag{8}$$

Let us introduce now two assumptions which will allow us to determine the probability density for the residence time T_r in the real mixer:
A1: The ideal residence time, T_i, and the time lag, T_p, are mutually independent.
A2: The mean degree of ideality of the real mixer, $\bar{z}$, is not a function of time.

With the aid of the assumption A1 we may describe the relation between the probability density of the random variable appearing in Eq. (5):

$$f_r(t) = \int_0^t f_i(t_i)\, f_p(t - t_i)\, dt_i . \tag{9}$$

In view of the fact that Z, according to Eq. (7), is a conditioned random variable (at the given value $T_r = t$), we may write an expression for its mean[3], $\bar{z}(t)$, which is not a function of time as assumed in A2:

$$\bar{z}(t) = \int_0^t t_i\, f_i(t_i)\, f_p(t - t_i)\, dt_r / t\, f(t) = a , \quad [0 \leqq a \leqq 1] . \tag{10}$$

The symbol a designates the value of the mean degree of ideality.
For some of the expressions to be presented subsequently it is convenient to introduce also the inverse of this quantity, b:

$$b = 1/a , \quad [1 \leqq b] . \tag{11}$$

Simultaneous solution of Eqs (9) and (10) yields the sought expression for the probability density $f_r(t)$. It is advantageous to make use for this purpose of the Laplace transform[7]. A function $f(t)$ transformed is designated by the symbol $\psi(p)$. From Eqs (10) and (11) then follows

$$-\frac{d\psi_r}{dp} = -b\, \frac{d\psi_i}{dp}\, \psi_p . \tag{12}$$

By transforming Eq. (9) we obtain

$$\psi_r = \psi_i \psi_p \tag{13}$$

and the transformation of Eq. (*2*) yields an explicit expression for ψ_i

$$\psi_i = \lambda/(p + \lambda) . \tag{14}$$

On differentiating Eq. (*14*), substituting into Eq. (*12*) and considering Eq. (*13*) we obtain the following differential equation

$$-\frac{d\psi_r}{dp} = \frac{b}{p + \lambda}\psi_r . \tag{15}$$

The appropriate boundary condition expresses the fact that $f_r(t)$ is the probability density and thus $\psi_{r|p=0} = 1$. Solution of Eq. (*15*) yields the expression (see the Appendix)

$$\psi_r = [\lambda/(p + \lambda)]^b \tag{15a}$$

and the inverse transformation gives the sought probability density[6]

$$f_r(t) = \lambda(\lambda t)^{b-1} \exp(-\lambda t)/\Gamma(b) \tag{16}$$

where $\Gamma(b)$ is the gamma function[9].

It is simple to calculate the mean residence time $\bar{t}_r$ in a real mixer by substituting from Eq. (*16*) into (*6*)

$$\bar{t}_r = b/\lambda = 1/\varkappa . \tag{17}$$

From Eqs (*3*), (*11*) and (*17*) there follows an expression for the ratio of the residence times in the "ideal" part of the mixer and the whole real mixer, this being equal to the degree of ideality

$$\bar{t}_i/\bar{t}_r = a .$$

Using a simple concept presuming that the residence time in the real mixer (as a random variable) is longer than the residence in its ideal part and using further two assumptions about the involved random variables we arrived at the sought gamma distribution.

It is apparent that for integer values of the parameter b Eq. (*16*) becomes identical with Eq. (*1*). Considering further the expression for the inverse value of the residence time, $\bar{t}_r$, from Eq. (*6*) and Eq. (*17*) we may substitute into (*16*) to get

$$f_r(t) = \varkappa\, b(\varkappa bt)^{b-1} \exp(-\varkappa bt)/\Gamma(b) \tag{18}$$

where the parameters b and $\varkappa$ are mutually independent.

For $b = 1$ Eq. (*18*) reduces to Eq. (*2*); for b growing to infinity an expression for plug flow results.

It can be shown (see the Appendix) that with the assumptions A1 and A2 the degree of ideality is a random variable with the probability density $f_z(Z)$, which is not a function of time

$$f_z(z) = (b - 1)(1 - z)^{b-2}, \quad [0 \leqq z \leqq 1]. \tag{19}$$

EXPERIMENTAL

Eq. (*18*) was tested on two experimental set-ups. The standard set-up consisted of a cylindrical vessel $0{\cdot}18\ m^3$ by volume equipped with an impeller with the blades inclined at an angle 45°. Principal characteristics of this set-up are given in Fig. 1 and the ranges of the experimental variables are indicated in Table I. The non-standard set-up was also a cylindrical vessel divided by horizontal perforated plated into four chambers (Fig. 2). In the center of each chamber there was a six-blade turbine impeller. These turbines were all of the same size and their blades were mounted on a common shaft. In one series of experimental runs the ratio of the diameter of the impeller to that of the vessel was 0·267; in the sedond series this ratio was 0·32. The ranges of the involved variables are also shown in Table I.

The charge of the mixed batch was in all cases water; the tracer was a saturated solution of potassium chloride injected at the inlet by a syringe in such a manner as to keep the duration

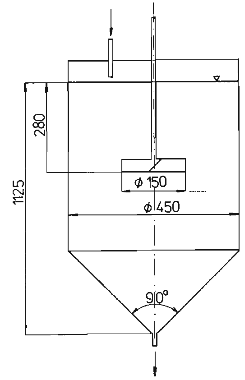

FIG. 1

Standard Experimental Set-Up Dimensions in mm.

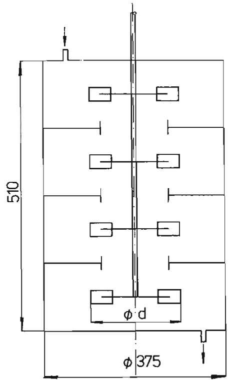

FIG. 2

Non-Standard Experimental Set-Up Dimensions in mm.

of the injection proper below 2 s. The input signal may thus be regarded as the δ-impulse. The response to this input was measured by conductometry. The temperature of the batch was held between 16 and 18°C and was recorded during experiments to the accuracy of 0·1°C. Some calibration tests were also carried out to find the temperature dependence of the conductivity of the solution. Because the output concentration fluctuates randomly, there were always at least five duplicate measurements carried out, the results of which were averaged. We had at our disposal also some results obtained on a large scale crystallizer with the volume of the batch 20 m^3 in which the response was measured by colorimetry[11].

The obtained time dependences of concentration were processed by a standardized routine consisting of finding first the normalized response as a function of time and the corresponding first and the second moments. By non-linear regression these results were processed to give the parameters $\varkappa$ and b in Eq. (*18*) for each series of experimental runs. As first estimates of these parameters we took the data found from the first and the second moments of the distributions.

In order that we might be able to at least roughly estimate to what extent the experimentally found distributions agree with the theoretical models we evaluated the corresponding deviations necessary for the F-test[12]. Fig. 3 shows the experimental data for an experimental run on the non-standard set-up, the corresponding gamma distribution and the same data for the industrial crystallizer.

The parameters $\varkappa$ and b were correlated in a similar manner as done by Conover[4], that means in dependence on experimental conditions characterized by the following dimensionless variables:

$$n/\varkappa = g_1(nV_r/\dot{V})^{h_1}\,, \tag{20}$$

$$b - 1 = g_2(n/\varkappa)^{h_2}\,, \tag{21}$$

where V_r is the volume of liquid in the batch, g and h are parameters which are functions of the geometrical arrangement of the equipment and probably also the physical properties of liquid. Their values were estimated by the least-squares method and summarized in Table I. This table show salso the recalculated data of Conover[4]. In a graphical form the correlation (*20*) is shown in Fig. 4; the correlation (*21*) is shown in Fig. 5. The dash-and-dot line corresponds to Conover's data[4].

Note: The scale on the ordinate of Fig. 62 in the cited paper[4] and the coefficient of the correlation equation are apparently two orders of magnitude higher; in the opposite case the experimentally found values of residence time in the mixer would be almost hundred times higher

TABLE I

Ranges of Experimental Conditions and Dependence of Parameters of the Gamma Distribution on these Conditions (Eq. (*20*) and (*21*))

Experimental arrangement	$\dot{V}$, 10^{-4} m^3 s^{-1}	n, s^{-1}	g_1	h_1	g_2	h_2
Standard	0·67—1·35	4·5—8·5	0·33	1·06	24·6	−0·502
Non-Standard	1·0 —2·3	0·8—5·0	1·63	0·885	55·1	−0·532
Conover's data[4]	1·85—4·35	0·7—7·7	0·47	1·002	150	−1·178

than the theoretical value resulting from the ratio of the volume of the equipment and the volumetric flow rate of liquid. The data in this sense were corrected. The parameter designated in the cited work[4] as is identical with our parameter λ and its inverse value thus is not, as erroneously stated in the work, equal to the mean residence time in the real mixer (Eq. (*17*)). In the cited paper, however, this deviation is not very large as Conover's experimental arrangement was rather close in behaviour to the ideal mixer (see Fig. 63 shown in that work).

DISCUSSION

Fig. 3 shows typical experimental data and it is apparent that the proposed function (*18*) well agrees with the experimentally found course. The results of the *F*-test also confirm this conclusion, even though they cannot serve in case of the function which is nonlinear in parameters as a strictly quantitative statistical criterion[13]. In any case though these results indicate that the gamma distribution with the parameter b describes better the experimentally found dependence than the cascade of ideal mixers with an integer m whose value deviates least from the calculated parameter b. It turns out that, as could be expected, the disagreement between the experimental data and the distribution with an integer parameter m is the greater the closer the value of b to unity.

Regarding the effect of experimental conditions on the parameters of distribution $\varkappa$ and b, our experiments further confirm the conclusions of Conover[4]. The residence

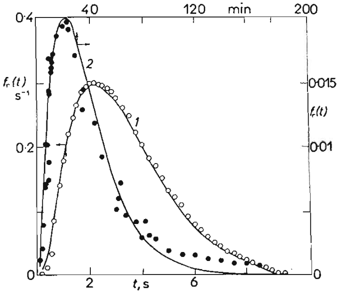

FIG. 3

Processing of Experimental Results

Non-standard set-up: proposed function *1*, exp. data ○; 20 m^3 crystallizer: proposed function *2*, exp. data ●.

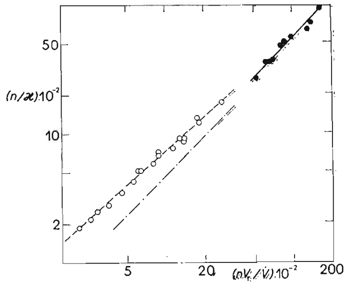

FIG. 4

Dependence of $\varkappa$ on experimental Conditions:

Standard set-up: correlation ———, exp. data ●. Non-standard set-up: correlation ------- exp. data ○. Conover's correlation[4]: correlation -.-.-.-.-, exp. data.

time in the real mixer $1/\varkappa$ (Eq. (6)) is practically directly proportional to the theoretical residence time $V_r/\dot{V}$ because the exponent h_1 in Eq. (20) for both experimental set-ups is close to unity. The behaviour of the real mixer approximates that of the ideal one (*i.e.* the parameter b is close to unity) as may be apparent from Eq. (21) with increasing rpms of the impeller, *i.e.* with increased intensity of mixing under otherwise identical conditions.

Both conclusions are plausible and fully confirm the suitability of the proposed model. Higher values of the parameter b for the set-up with four impellers (Fig. 5) agree well with the concept of the cascade of ideal mixers; at higher rpms there exist apparently an intensive flow of liquid through the perforated plates and individual chambers cannot be regarded as isolated mixers.

The deviations of our data from those of Conover[4] in case of the parameter b may be probably accounted for by increased sensitivity of this parameter to the geometrical arrangement of the set-up.

While the physical meaning of the parameter $\varkappa$ as the inverse of the mean residence time is clear it is not the case of the parameter b as long as we are not prepared to accept the rather dubious character of non-integer ideal mixer. As an alternative explanation we offer the following:

We shall assume that in a real mixer of volume V_r there is an ideally mixed core of volume V_i (Fig. 6) and, further, that the residence time within and without the

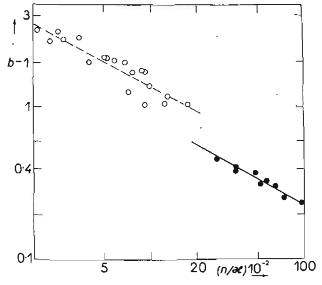

FIG. 5

Dependence of b on Experimental Conditions

The same caption as that for Fig. 4.

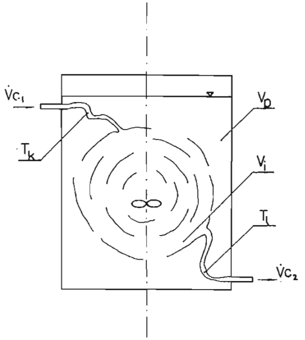

FIG. 6

Scheme of Real Mixer

V_i ideally mixed core, V_p region of plug flow, T_k inlet stream tube, T_1 outlet stream tube.

core are proportional to the corresponding volumes. With respect to Eq. (*7*) we thus may write

$$T_i/t = V_i(t)/V_r = Z(t) \tag{22}$$

and the above defined random function of time may be thought of as the volume of the ideally mixed core relatively to the volume of the whole mixer. This relative volume varies randomly in time and, according to Eq. (*19*), is a stationary random function. Now the question is that of the character of the flow outside the ideal core, *i.e.* in the volume designated by V_p (Fig. 6). Thus clearly, $V_r = V_i + V_p$. Let us consider that at a time $t = 0$ we inject a tracer of concentration $c_1(t)$ in a manner defined above (see the test between Eqs (*3*) and (*4*)) and write down the balance over the real mixer

$$\dot{V}c_1 - \dot{V}c_2 = V_i\,\varphi_i(c_2) + V_p\,\varphi_p(c_2) \tag{23}$$

where for the moment unknown functions φ_i and φ_p characterize the flow within and without the ideally mixed core respectively. For the core we may clearly write (see also the Appendix)

$$\varphi_i(c_2) = \mathrm{d}c_2/\mathrm{d}t\,. \tag{24}$$

Let us assume that within the volume V_p the liquid moves at plug flow. It can be shown then that (see the Appendix)

$$\varphi_p(c_2) = -c_2/t\,. \tag{25}$$

Substituting these relations into Eq. (*23*) and dividing by V_r we obtain

$$\varkappa c_1 - \varkappa c_2 = Z\,\mathrm{d}c_2/\mathrm{d}t + (1 - Z)(-c_2/t)\,.$$

Finally the whole equation is averaged with respect to Z by multiplying the equation by the probability density $f_z(z)$ defined by Eq. (*19*) and integrating between the limits 0 and 1. Considering Eq. (*10*) we thus obtain the following differential equation:

$$a\,\mathrm{d}c_2/\mathrm{d}t + (1 - a)(-c_2/t) = \varkappa c_1 - \varkappa c_2\,. \tag{26}$$

A solution of this equation (see the Appendix) is the expression

$$c_2(t) = \alpha\,f_r(t)\,. \tag{27}$$

This suggests that the residence time in the real mixers possesses the gamma distribution.

The whole real mixer may thus be looked upon as a superposition of the ideal mixer and plug flow while the superposition in this context does not mean a mere summation of the residence times T_i and T_p. For this reason we shall present here a rather imperfect concept based on the combination of the ideal mixer and plug flow: Consider first a stream tube, T_k, of small cross sectional area ΔS_j through which a small portion of the input impulse enters at the instant $t = 0$ the real mixer. This stream tube of the length k_j empties into the ideally mixed core of constant volume V_i. Consider another stream tube, T_f, of the same cross section and the length f_j which emerges from the mixed core and empties at the discharge from the real mixer (Fig. 6). Let us assume that the liquid in both stream tubes is at plug flow velocity v. For the probability density of the overall residence time of liquid passing through both tubes and the ideally mixed core we may then write

$$f_{rj}(t) = \int_0^t \delta(t - t_2 - l_j/v) \int_0^{t_2} \varkappa b \exp\left[-\varkappa b(t_2 - t_1)\right] \delta(t_1 - k_j/v)\, dt_1\, dt_2 \quad (28a)$$

from which after integration we obtain

$$f_{rj}(t) = \begin{cases} \varkappa b \exp\left[-\varkappa b\left(t - \dfrac{l_j + k_j}{v}\right)\right], & \left[t \geqq \dfrac{l_j + k_j}{v}\right] \\ 0, & \left[t < \dfrac{l_j + k_j}{v}\right] \end{cases} \quad (28b)$$

while $b = V_r/V_i$.

Let us consider further that there are many such tubes and that the input impulse is uniformly distributed among them. If at the outlet the content of these tubes in each instant is perfectly mixed then all stream tubes may be schematically depicted as straight tubes with that the sequence of the ideal mixer and the stream tube in the outlet part of the real mixer was interchanged (Fig. 7*a*).

Further we define the total cross sectional area of all stream tubes S by

$$S = \sum_{j=1}^{N} \Delta S_j\,.$$

Here N is the number of all tubes and the quantity L is defined by $L = V_r/S$, which is the mean length of the tubes assuming that these fill the whole volume of the real mixer (plug flow in the mixer). Finally we can write an expression for the velocity of liquid, v, assuming that v is constant in all stream tubes and equal $\dot{V}/S$, and an expression for the residence times, t_{pj} in a stream tube, considering Eq. (*28*) and the just written definitions:

$$t_{pj} = (l_j + k_j)/v = [(l_j + k_j)\, V_r/LS]\,(S/\dot{V}) = x_j/\varkappa \qquad (29)$$

where $x_j = (l_j + k_j)/L$ is the dimensionless length of the stream tube.

The probability density $f_r(t)$ of the residence time in the whole real mixer is obtained as a sum over all stream tubes weighted by their cross sections after we had substituted from Eq. (*29*) into (*28*)

$$f_r(t) = (\varkappa b)/S \sum_{j=1}^{N} \exp\left[-\varkappa b(t - t_{pj})\right] \Delta S_j\,. \qquad (30)$$

The expression on the right hand side of Eq. (*30*) describes the function of the ideal mixer in Fig. *7a* into which at the instant $t = t_{pj}$ we inject a new dose of the tracer. From Eq. (*28*) and Fig. 7 it is apparent that following the sign for summation the non-zero terms will be only those corresponding to sufficiently short stream tubes, *i.e.* such for which t_{pj} is shorter than the interval t from the instant of the entry of the impulse into the real mixer. The number of the stream tubes through which the tracer has passed until the time t into the ideal mixer is thus a function of this time instant.

If the number N of the stream tubes is large and the cross section ΔS_j of each of them is sufficiently small, as may be anticipated in our case, the summation in Eq.

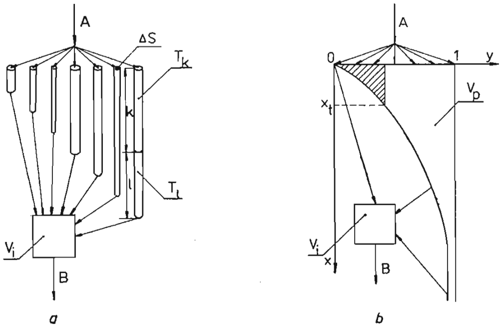

FIG. 7

Gamma Distribution of Residence Time in Real Mixer

a) Individual stream tubes; *b*) relative length of stream tubes after limiting transition; A inlet of the real mixer; B outlet of the real mixer; the remaining caption is the same asthat for Fig. 6.

(*30*) may be replaced by integration and we may write

$$f_r(t) \doteq \varkappa b \int_0^1 \exp\left[-\varkappa b(t - t_p)\right] \mathrm{d}y\,, \tag{31}$$

where $\Delta y_j = \Delta S_j/S$.

Now we exchange the variable in Eq. (*31*); the limits of the newly formed integral can be determined by considering Eq. (*30*). Thus

$$f_r(t) = \varkappa b \int_0^t \exp\left[-\varkappa(t - t_p)\right] (\mathrm{d}y/\mathrm{d}x)\,(\mathrm{d}x/\mathrm{d}t_p)\,\mathrm{d}t_p\,. \tag{32}$$

While the second derivative on the right hand side can be easily found from Eq. (*29*) ($\mathrm{d}x/\mathrm{d}t_p = \varkappa$), for the calculation of the first derivative we have to know explicitly the form of the function $y = y(x)$, *i.e.* the curve which confines the lower end of the stream tube in Fig. 7. This function must clearly satisfy the following two stipulations: i) The solution of the integral in Eq. (*32*) must be the Eq. (*18*), ii) the volume of all stream tubes must equal the volume of the real mixer minus the volume of the ideal part. This means that

$$\int_0^1 x\,\mathrm{d}y = 1 - a\,. \tag{33}$$

Such a function exists, *i.e.*

$$y(x) = \{\gamma(xb;\, b - 1)\}/\Gamma(b - 1) \tag{34}$$

as we may find by substituting from Eq. (*34*) into (*32*) and (*33*) and by considering Eq. (*11*). The expression in the numerator is the incomplete gamma function[10], defined by

$$\gamma(xb;\, b - 1) = \int_0^{xb} \exp\left(-u\right) u^{b-2}\,\mathrm{d}u\,.$$

The course of this function is depicted in Fig. 7*b*. The shaded area designates the stream tubes through which by the instant t the inlet impulse will have reached the ideal mixer; the length of the longest of these tubes is x_t.

The above concept suggests that the total length of a single stream tube is zero: the ideal mixer must – as could be expected – be "in contact" with the inlet and the discharge of the mixer.

The concept presented here has numerous shortcomings (*e.g.* infinite stream tube, constant velocity of liquid in all stream tubes) but in spite of this is capable of accounting for the character of the flow in a real mixer, leading to the gamma distribution of the residence times. It is believed that more plausible models can be found. In our case more attention should be paid to the description of the residence times the prospects of which has confirmed our experiments.

APPENDIX

The Derivation of the Probability Density for the Degree of Ideality of the Mixer

Consider two random variables T_i and T_r, the probability density of which, $f_i(t_i)$ and $f_r(t)$, are given by Eqs (*2*) and (*16*). Let us define two more random variables

$$T_p = T_r - T_i \tag{35}$$

(Eq. (*5*)) and

$$Z = T_i/(T_i + T_p) . \tag{36}$$

Now we seek the probability density, $f_p(t_p)$, for the random variable T_p, and the conditioned probability density, $f_z(z \mid t)$, of the random variable Z for $T_r = t$ (Eq. (*7*)).

In the first part of the problem we shall start from Eq. (*9*) and its Laplace transform (*13*). After substituting from Eqs (*14*) and (*15a*) into Eq. (*13*) and some algebraic manipulations we obtain the expression

$$\psi_p = [\lambda/(p + \lambda)]^{b-1} ,$$

the object of which is[5]

$$f_p(t_p) = \lambda(\lambda t_p)^{b-2} \exp(-\lambda t_p)/\Gamma(b - 1) . \tag{37}$$

For the solution of the second problem we shall find first the probability density of the variables Z and T_r using the known probabilities $f_i(t_i)$, $f_p(t_p)$:

$$f_{zt}(t, z) = f_i[t_i(t, z)] \, f_p[t_p(t, z)] \left| \frac{D(t_i, t_p)}{D(t, z)} \right| . \tag{38}$$

At the extreme right of the last equation there is the symbol for the absolute value of the Jacobi's determinant of the transformation of the variables[2]

$$t_i = zt , \quad t_p = (1 - z) \, t .$$

In view of Eqs (*2*) and (*37*) we obtain

$$f_{zt}(t, z) = \lambda \exp(-\lambda tz) \, \lambda t(1 - z))^{b-2} \exp(-\lambda t(1 - z)) \, t/\Gamma(1 - b) , \tag{39}$$

while, as may be readily apparent

$$\left| \frac{D(t_i, t_p)}{D(t, z)} \right| = t .$$

Finally we write down the expression for the sought conditioned probability density[3]

$$f_z(z \mid t) = f_{zt}(t, z)/f_t(t)\,. \tag{40}$$

After substituting from Eqs (*16*) and (*39*) into Eq. (*40*) we can easily verify that

$$f_z(z \mid t) = f_z(z) = (b - 1)(1 - z)^{b-2}\,.$$

This way we arrived at Eq. (*19*) from which we may conclude that the conditioned probability density is not a function of time t and that the random function $Z(t)$ in Eq. (*7*) is a stationary random function.

Solution of the Differential Equation for the Real Mixer by Laplace Transform

The Laplace transform[8] of Eq. (*26*) is

$$ap\psi_c - (1 - a)\int_p^\infty \psi_c \,\mathrm{d}u = \varkappa\alpha - \varkappa\psi_c \tag{41}$$

where ψ_c is the image of the function $c_2(t)$ and the constant α is the image of the function $c_1(t)$. By differentiation of Eq. (*41*) we obtain

$$a\psi_c + ap(\mathrm{d}\psi_c/\mathrm{d}p) + (1 - a)\,\psi_c = -\varkappa(\mathrm{d}\psi_c/\mathrm{d}p)\,. \tag{42}$$

After some algebraic rearrangement of Eq. (*42*) and considering Eq. (*11*) we arrive at the following ordinary differential equation

$$\mathrm{d}\psi_c/\mathrm{d}p = -b\psi_c/(p + b\varkappa) \tag{43}$$

with the boundary condition $\psi_{c|p=0} = \alpha$. After separation of variables

$$\mathrm{d}\psi_c/\psi_c = -b\,\mathrm{d}p/(p + \varkappa b)$$

the solution takes the form

$$\ln \psi_c = -b \ln (p + \varkappa b) + \ln C$$

and hence

$$\psi_c = C/(p + \varkappa p)^b\,.$$

From the boundary condition we obtain an expression for the constant C in the form $C = \alpha(\varkappa b)^b$. Thus the final solution of Eq. (*43*) is

$$\psi_c = \alpha[\varkappa b/(p + \varkappa b)]^b\,.$$

For $\varkappa b = \lambda$ (Eq. (*17*)) and $\alpha = 1$ we obtain Eq. (*15a*). The inverse transformation of Eq. (*44*) yields

$$c_2(t) = \alpha[\varkappa b(\varkappa bt)^{b-1} \exp(-\varkappa bt)]/\Gamma(b)\,. \tag{45}$$

Comparing Eqs (*45*) and (*18*) we obtain Eq. (*27*).

a) Omitting the second term on the left hand side of Eq. (*42*) we obtain the solution

$$\psi_c = \alpha[\varkappa b/(\varkappa b + p)]\,. \tag{46}$$

The inverse transformation yields

$$c_2(t) = \alpha\varkappa b \exp\left(-\varkappa bt\right)$$

which confirms the statement expressed by Eq. (*24*).

b) Omitting the first term on the left hand side of Eq. (*42*), differentiating with respect to p and dividing by the constant a we obtain

$$(b - 1)\,\psi_c = -\varkappa b\, \mathrm{d}\psi_c/\mathrm{d}p\,. \tag{47}$$

The boundary condition for this equation is clearly the same as that for Eq. (43). A solution of Eq. (*47*) is thus

$$\psi_c = \alpha \exp\left(-\frac{b - 1}{\varkappa b}\, p\right) \tag{48}$$

whose object is the δ-function[8]

$$c_2(t) = \alpha\delta[t - (b - 1)/\varkappa b] \tag{49}$$

which proves Eq. (*25*) and the statement preceding this equation.

LIST OF SYMBOLS

a	mean degree of ideality
b	inverse value of the mean degree of ideality
C	integration constant, kg s m^{-3}
c_1	inlet concentration, kg m^{-3}
c_2	outlet concentration, kg m^{-3}
D	diameter of vessel, m
d	diameter of impeller, m
$E(\,)$	operator of expected value (mean)
F	distribution function
f	probability density
f_z	probability density of variable z
g_1	coefficient in Eq. (*20*)
g_2	coefficient in Eq. (*21*)
h_1	exponent in Eq. (*20*)
h_2	exponent in Eq. (*21*)
k	length of stream tube emerging from the ideally mixed core, m
L	mean length of the stream tube, m
l	length of the stream tube before entering the ideally mixed core, m
m	number of ideal mixers

N number of stream tubes
n frequency of revolution of the impeller, s^{-1}
$P\{\ \}$ probability
p Laplace variable, s^{-1}
S total cross sectional area of all stream tubes, m^2
ΔS cross sectional area of a stream tube, m^2
T residence time (random variable), s
t time, s
u integration variable
V volume of mixer, m^3
$\dot{V}$ volumetric flow rate of liquid, $m^3 s^{-1}$
v velocity of liquid in stream tube, $m s^{-1}$
x relative (dimensionless) length of stream tube
y relative cross section of stream tubes
Z degree of ideality of real mixer (random variable – relative volume of the ideally mixed core in the real mixer)
z degree of ideality of the real mixer
α proportionality constant (amount of tracer injected), $kg s^{-1} m^{-3}$
Γ gamma function
γ incomplete gamma function
σ Dirac function, s^{-1}
φ characteristic of flow in mixer, $kg m^{-3} s^{-1}$
ψ image in the Laplace transform space
ψ_c image of function $c_2(t)$ in the Laplace transform space, $kg s m^{-3}$
λ parameter of distribution, s^{-1}
$\varkappa$ inverse quantity to the residence time in mixer, s^{-1}

SUBSCRIPTS

i related to ideal mixer
j related to j-th stream tube
p related to plug flow
r related to real mixer

REFERENCES

1. Feller W.: *An Introduction to Probability Theory and its Application* II, p. 66. Izd. Mir, Moscow 1967.
2. See ref. 1, p. 91.
3. See ref. 1, p. 95 and 98.
4. Conover J. A.: *Thesis*. St. Louis University, 1960. Cited from: (W. V. Uhl, J. B. Gray, Eds): *Mixing* I, p. 267. Academic Press, New York 1966.
5. Pírko Z., Veith J.: *Laplaceova transformace*, p. 36. SNTL, Prague 1972.
6. See ref. 5, p. 106.
7. See ref. 5, p. 145.
8. See ref. 5, 38 and 153.
9. Gradstein J. S., Ryshik J. M.: *Tablicy Integralov, Summ, Riadov i Proizvedenii*, p. 947. Gos. Izd. Fiz.-Mat. Lit., Moscow 1963.
10. See ref. 9, p. 954.

11. Havránek J.: Paper Presented at the Conference on Industrial Crystallization, Ústí nad Labem 1974.
12. Himmelblau D. M.: *Process Analysis by Statistical Methods*, p. 176. Wiley, New York 1970.
13. See ref. 12, p. 28.
14. Wen C. Y., Fan L. T.: *Models for Flow Systems and Chemical Reactors*, p. 281. Dekker, New York 1975.
15. Cha L. C.: *Thesis*. Kansas State University, 1965. Cited from ref. 14.
16. Johnson J. L.: *Thesis*. Kansas State University, 1966. Cited from ref. 14.
17. Johnson J. L., Fan L. T., Wu Y. S.: Ind. Eng. Chem., Process Des. Develop. *10*, 425 (1971).

Translated by V. Staněk.